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Author(s):

JAMES D. LOUCK, T-7, THEORETICAL DIVISION

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MODIFIED GELFAND-TSETLIN PATTERNS, LATTICE PERMUTATIONS, AND SKEW-TABLEAU POLYNOMIALS

James D. Louck
Los Alamos National Laboratory, Theoretical Division
Los Alamos, NM 87545 USA

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A modification of the well-known Gelfand-Tsetlin patterns, which are one-to-one with Young-Weyl standard tableaux is introduced. These new patterns are in one-to-one correspondence with skew-tableaux, and with a slight modification can be used to enumerate lattice permutations. In particular, the coupling rule for angular momentum takes an elementary form in terms of these modified patterns. These interrelations will be presented, together with an outline of the construction of a class of polynomials that generalizes the skew Schur functions.

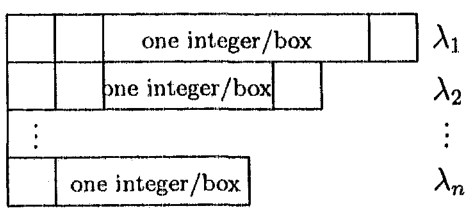
1 Introduction and Review of Combinatorial Concepts

It is well known that semistandard Young-Weyl tableau of shape $\lambda \in \mathbb{P}ar_n$, where $\mathbb{P}ar_n$ denotes the set of all partitions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n), \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0$, having n parts, counting 0 as a part, is bijective with the set of Gelfand-Tsetlin patterns of shape λ. The purpose of this article is to show how this result generalizes to a bijection between semistandard skew tableau of shape $\lambda - \mu$ and skew Gelfand-Tsetlin patterns of shape $\lambda - \mu$, structures that are defined below. This result, in turn, allows the generalization of the well known $D^\lambda-$polynomials[1-4] to skew $D^{\lambda/\mu}-$polynomials. An consequence of this generalization is that the trace of the skew $D^{\lambda/\mu}-$polynomials yields the skew Schur functions $s_{\lambda/\mu}$, just as the trace of the ordinary $D^\lambda-$polynomials yields the ordinary Schur functions s_λ. We also formulate the concept of a modified Gelfand-Tsetlin pattern, and show their relation to lattice permutations and Littlewood-Richardson numbers. The properties of Littlewood-Richardson numbers are very important for the study of composite holistic multiparticle quantum systems to which the $D^\lambda-$polynomials and the $D^{\lambda/\mu}-$polynomials have applications through multiple Kronecker products of such polynomials and their reduction into irreducible forms. We do not address the latter in this article, but instead review and set forth the combinatorial approaches to the subject.

The interest in formulating tableau results in terms of Gelfand-Tsetlin patterns originates from the Weyl[5] group-subgroup significance of the conditions associated directly with Gelfand-Tsetlin[6] patterns, and the subsequent use of such patterns in numerous physical applications. We begin by a review of several well known results.[7]

2.1 Semistandard Young-Weyl Tableaux and Gelfand-Tsetlin Patterns

Semistandard Young-Weyl tableau (SSYW): A partition $\lambda \in \mathbb{P}ar_n$ is sometimes called a *shape*.A Young-Weyl tableau is a shape $\lambda \in \mathbb{P}ar_n$ in which the integers $1, 2, \ldots, n$ are distributed among the $|\lambda| = \lambda_1 + \cdots + \lambda_n$ boxes, one in each box, according to the rules:



$\rightarrow$ weakly increasing across each row

$\downarrow$ strictly increasing down each column

Notations:

$\mathbb{T}_\lambda =$ set of all semistandard tableau of shape λ,
$\alpha =$ weight of semistandard tableau $= (\alpha_1, \ldots, \alpha_n), \alpha_i =$number of *i's*,
$\mathbb{T}_\lambda(\alpha) =$ set of all semistandard tableau of shape λ and weight α,
$\mathbb{W}_\lambda =$ set of all weights of semistandard tableaux of shape λ,
$K(\lambda, \alpha) =$ multiplicity of a weight $\alpha \in \mathbb{W}_\lambda$.

Gelfand-Tsetlin (GT) pattern: Let $\rho \in \mathbb{P}ar_r, \sigma \in \mathbb{P}ar_{r-1}, 1 < r \leq n$. The notation $\rho \succ \sigma$ means that the pair of partitions ρ and σ satisfy the betweenness conditions:

$$\rho_1 \geq \sigma_1 \geq \rho_2 \geq \sigma_2 \geq \cdots \geq \sigma_{r-1} \geq \rho_r \geq 0.$$

Two-rowed presentation:

$$\begin{pmatrix} \rho_1 & & \rho_2 & & \rho_3 & & \cdots & & \rho_{r-1} & & \rho_r \\ & \sigma_1 & & \sigma_2 & & & \cdots & & & \sigma_{r-1} & \end{pmatrix}.$$

The geometric placement of numbers is intended to suggest "betweenness." A GT pattern of shape λ is a sequence of n partitions $\lambda, \tau, \cdots \nu, \mu$ satisfying

$$\lambda \in \mathbb{P}ar_n, \tau \in \mathbb{P}ar_{n-1}, \cdots \nu \in \mathbb{P}ar_2, \mu \in \mathbb{P}ar_1,$$

$$\lambda \succ \tau \succ \cdots \succ \nu \succ \mu.$$

A GT pattern of shape λ may be presented as triangular array with n rows:

$$\begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_{n-1} & \lambda_n \\ \tau_1 & \tau_2 & \cdots & \tau_{n-1} & \\ & & \vdots & & \\ & \nu_1 & \nu_2 & & \\ & & \mu_1 & & \end{pmatrix}, \ \lambda \succ \tau \succ \cdots \succ \nu \succ \mu.$$

The geometric placement of these partitions is intended to indicate "betweenness."

Notations:

$\mathbb{G}_\lambda =$ set of all GT patterns of shape $\lambda \in \mathbb{P}ar_n$,
$\alpha = (|\mu|, |\nu| - |\mu|, \ldots, |\lambda| - |\tau|)$,
$\mathbb{G}_\lambda(\alpha) =$ set of all GT patterns of weight α ,
$\mathbb{W}_\lambda =$ set of all weights of GT patterns of shape λ,
$K_(\lambda, \alpha) =$ multiplicity of a weight $\alpha \in \mathbb{W}_\lambda$.

The significant result is the bijection between the set of semistandard tableaux of shape λ and the set of GT patterns of shape λ of the same weight,

$$\mathbb{T}_\lambda(\alpha) \overset{BIJECTION}{\longleftrightarrow} \mathbb{G}_\lambda(\alpha),$$

as given by the following rules:

$$\begin{array}{|l|} \hline \lambda_1 \text{ integers} \\ \hline \lambda_2 \text{ integers} \\ \hline \vdots \\ \hline \lambda_n \text{ integers} \\ \hline \end{array} \quad \longleftrightarrow \quad \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_{n-1} & \lambda_n \\ \tau_1 & \tau_2 & \cdots & \tau_{n-1} & \\ & & \vdots & & \\ & \nu_1 & \nu_2 . & & \\ & & \mu_1 & & \end{pmatrix}$$

mapping rule: The shapes (partitions) in the GT pattern are obtained by sequential removal of integers from the semistandard tableau:

λ is the shape of the tableau;

τ is the shape of the tableau obtained by removing all n's;

$\vdots$

ν is the shape of the tableau obtained by removing all 3's;
μ is the shape of the tableau obtained by removing all 2's;

inverse rule: The semistandard tableau is obtained from the GT pattern by the rule

insert 1's in the shape μ;
followed by 2's in the new boxes contained in shape ν and not in shape μ;

$\vdots$

followed by n's in the new boxes contained in shape λ and not in shape τ.

Example:

The relation between $SSYW$ tableaux and GT patterns of the same shape is illustrated for $\lambda = (6, 5, 3, 0)$ by

$$\begin{array}{|c|c|c|c|c|c|}\hline 1 & 1 & 1 & 2 & 3 & 4 \\ \hline 2 & 2 & 3 & 3 & 4 \\ \cline{1-5} 3 & 4 & 4 \\ \cline{1-3}\end{array} \quad \longleftrightarrow \quad \begin{pmatrix} 6 & & 5 & & 3 & & 0 \\ & 5 & & 4 & & 1 & \\ & & 4 & & 2 & & \\ & & & 3 & & & \end{pmatrix}$$

The bijection described above extends, of course, to the full sets of semistandard Young-Weyl tableaux and Gelfand-Tsetlin patterns of shape λ :

$$\mathbb{T}_\lambda \overset{BIJECTION}{\longleftrightarrow} \mathbb{G}_\lambda .$$

2 The D^λ–Polynomial

One of the significant applications of SSYW tableaux,or, equivalently, of GT patterns is their role as indexing sets or labels of a class of polynomials that arise in many different contexts. These polynomials have been discussed in a number of articles.[1–3,7] Some of their principal properties are recalled here for the purpose of showing how they recur in the context of skew GT and modified GT patterns in Sections 3 and 6.2 .

2.1 *Basic Orthogonal Maclaurin Polynomials*

We adopt the following notations and definitions:

$$\frac{Z^A}{A!} = \prod_{i,j=1}^{n} \frac{z_{ij}^{a_{ij}}}{a_{ij}!}$$

$$\text{variables} = Z = \begin{pmatrix} z_{11} & z_{12} & \cdots & z_{1n} \\ z_{21} & z_{22} & \cdots & z_{2n} \\ & & \vdots & \\ z_{n1} & z_{n2} & \cdots & z_{nn} \end{pmatrix}$$

$$
\text{exponents} = A = \begin{array}{c} \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & & \vdots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{matrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{matrix} \\ \begin{matrix} \alpha'_1 & \alpha'_2 & \cdots & \alpha'_n \end{matrix} \end{array}
$$

$$
\begin{aligned} M^p_{n\times n}(\alpha,\alpha') = \Big\{ & n \times n \text{ arrays } A \,\Big| \\ & \text{row sums} = \alpha, \text{ column sums} = \alpha', |\alpha| = |\alpha'| = p \Big\} \end{aligned}
$$

The Maclaurin polynomials are homogeneous as follows:

$$
\begin{aligned} \alpha_i &= \sum_{j=1}^{n} a_{ij} \text{ in } z_i = (z_{i1}, z_{i2}, \ldots, z_{in}) = \text{row } i \text{ of } Z, \\ \alpha'_j &= \sum_{i=1}^{n} a_{ij} \text{ in } z^j = (z_{1j}, z_{2j}, \ldots, z_{nj}) = \text{column } j \text{ of } Z. \end{aligned}
$$

The Maclaurin polynomials are orthogonal in the inner product (,) described in detail by Louck.[3]

$$
(Z^A, Z^B) = \delta(A,B)A!.
$$

Basic Orthogonal D^λ–Polynomials

We now define a class of invertible real linear transformations of the Z^A polynomials that preserve their homogeneity properties in the rows and columns of the variable matrix Z :

$$
\begin{aligned} D\begin{pmatrix} m' \\ \lambda \\ m \end{pmatrix}(Z) &= \sum_{A \in M^p_{n\times n}(\alpha,\alpha')} C\begin{pmatrix} m' \\ \lambda \\ m \end{pmatrix}(A)\frac{Z^A}{A!}, \\ \frac{Z^A}{A!} &= \sum_{\substack{|\lambda|=p \\ m,m' \in \mathbb{G}_\lambda(\alpha,\alpha')}} \frac{1}{M(\lambda)A!} C\begin{pmatrix} m' \\ \lambda \\ m \end{pmatrix}(A) D\begin{pmatrix} m' \\ \lambda \\ m \end{pmatrix}(Z). \end{aligned}
$$

The notations in this definition are:

1. $\binom{\lambda}{m}$ is a Gelfand-Tsetlin GT pattern, and $\binom{m'}{\lambda}$ is an inverted GT pattern. We now write a GT pattern in the more detailed form

$$
\begin{pmatrix} \lambda \\ m \end{pmatrix} = \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ m_{1,n-1} & m_{2,n-1} & \cdots & m_{n-1,n-1} \\ & \vdots & & \\ m_{1,2} & m_{2,2} & & \\ m_{1,1} & & & \end{pmatrix},
$$

in which row j is given by

$$m_j = (m_{1,j}, m_{2,j}, \cdots, m_{j,j}) \in \mathbb{P}ar_j, j = 1, 2, \ldots, n, \lambda = m_n.$$

2. α, α' are weights of the GT patterns:

$$\alpha = W\binom{\lambda}{m} = (\alpha_1, \alpha_2, \ldots, \alpha_n),\ \alpha' = W\binom{\lambda}{m'} = (\alpha'_1, \alpha'_2, \ldots, \alpha'_n).$$

3. $\mathbb{G}_\lambda(\alpha, \alpha')$ is the set of double GT patterns of weight (α, α') and partition λ with cardinality

$$|\mathbb{G}_\lambda(\alpha, \alpha')| = K(\lambda, \alpha)K(\lambda, \alpha').$$

4. $M(\lambda)$ is the invariant normalizing factor defined by

$$M(\lambda) = \left(\prod_{i=1}^{n} (\lambda_i + n - i)!\right) /1!2! \cdots (n-1)! Dim\lambda,$$

in which $Dim\lambda$ is the Weyl dimension formula:

$$Dim\lambda = |\mathbb{G}_\lambda| = \left(\prod_{1 \le i < j \le n} (\lambda_i - \lambda_j + j - i)\right) /1!2! \cdots (n-1)!.$$

The orthogonality and normalization of the D^λ-polynomials are expressed by

$$\left(D\begin{pmatrix} m''' \\ \lambda \\ m'' \end{pmatrix}(Z),\ D\begin{pmatrix} m' \\ \lambda' \\ m \end{pmatrix}(Z) \right) = \delta_{m,m''}\delta_{m',m'''}\delta_{\lambda,\lambda'}M(\lambda).$$

2.3 Combinatorial Definition of the $C-$Coefficients

$$\left(D\begin{pmatrix} m''' \\ \lambda + e_\tau \\ m'' \end{pmatrix}(Z), z_{ij}D\begin{pmatrix} m' \\ \lambda \\ m \end{pmatrix}(Z) \right)$$
$$= \left\langle \begin{matrix} \lambda + e_\tau \\ m'' \end{matrix} \;|\, t_{i\tau} \,|\; \begin{matrix} \lambda \\ m \end{matrix} \right\rangle \left\langle \begin{matrix} \lambda + e_\tau \\ m''' \end{matrix} \;|\, t_{j\tau} \,|\; \begin{matrix} \lambda \\ m' \end{matrix} \right\rangle$$

$e_\tau =$ unit row vector of length $n = (0, \ldots, 0, 1, 0, \ldots, 0), 1$ in position τ

The n^2 quantities $t_{i\tau}$ are called fundamental shift operators. The coefficients

$$\left\langle \begin{matrix} \lambda + e_\tau \\ m'' \end{matrix} \;|\, t_{i\tau} \,|\; \begin{matrix} \lambda \\ m \end{matrix} \right\rangle, i, j = 1, 2, \ldots n$$

are fully defined as functions over arc digraphs (earlier called the pattern calculus[8]). The above matrix element relation and the arc digraphs determine uniquely the $D^\lambda-$polynomials. It is the algebra of these fundamental *shift operators that places the construction of the $C-$coefficients, hence, the $D^{\lambda}-$polynomials on a fully combinatorial basis.*[3]

For the purposes of this article, we henceforth take the $D^{\lambda}-$polynomials as fully known. It is useful to summarize some of their principal properties.

2.4 Summary of Structural Properties of the $D^{\lambda}-$Matrices

Matrix form of basic polynomials:

$$D^{\lambda}(Z) = \sum_{A \in M^{p}_{n\times n}} \frac{Z^{A}}{A!} C^{\lambda}(A), \text{each } \lambda \in \mathbb{P}ar_{n}$$

$$M^{p}_{n\times n} = \bigcup_{\alpha,\alpha' \in \mathbf{W}_{\lambda}} M^{p}_{n\times n}(\alpha,\alpha').$$

In this relation, $D^{\lambda}(Z)$ and $C^{\lambda}(A)$ are matrices of dimension $Dim(\lambda)$ given by the Weyl dimension formula, and the coefficients in this matrix relation are the Maclaurin monomials $Z^{A}/A!$. The following properties offer but a glimpse at the important structural properties of these matrices, whose elements are the polynomials

$$D\begin{pmatrix} m' \\ \lambda \\ m \end{pmatrix}(Z),$$

whose rows and columns are enumerated by the Gelfand-Tsetlin patterns $\binom{\lambda}{m}$ and $\binom{\lambda}{m'}$.

1. Multiplication of $D^{\lambda}(Z)$ matrices:

$$D^{\lambda}(X)D^{\lambda}(Y) = D^{\lambda}(XY),$$

for arbitrary matrices X and Y. A full combinatorial proof of this relation can be given, which starts with the proof given in [Chen], and then uses Pieri's rule for multiplying certain of these polynomials.

2. Multiplication of $C^{\lambda}(A)$ matrices:

$$C^{\lambda}(A)C^{\lambda}(B) = \sum_{C \in M^{p}_{n\times n}(\alpha,\gamma)} \left\{ \begin{matrix} C \\ A\,B \end{matrix} \right\} C^{\lambda}(C),$$

where

$$A \in M^{p}_{n\times n}(\alpha,\beta), B \in M^{p}_{n\times n}(\beta,\gamma),$$

and the structure constants are given by the double inner product of Maclaurin polynomials [Italy].

$$\left\{ \begin{matrix} C \\ A\,B \end{matrix} \right\} = \left(X^{A}, \left(Y^{B}, (XY)^{C}\right)\right)/C!.$$

3. Matrix Schur function:

$$D^{\mu}(Z) \otimes D^{\nu}(Z) \sim \sum_{\lambda} c^{\lambda}_{\mu\nu} D^{\lambda}(Z),$$

in which the coefficients $c^{\lambda}_{\mu,\nu}$ are the Littlewood-Richardson numbers.

4. Diagonal properties:

$$D^{\lambda}(diag(x_1, x_2, \ldots, x_n)) = \sum_{\alpha \in \mathbf{W}_{\lambda}} \oplus x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} I_{K(\lambda,\alpha)},$$

$$I_{K(\lambda,\alpha)} = \text{ identity matrix of dimension } K(\lambda,\alpha),$$

$$D^{\lambda}(I_n) = I_{Dim\lambda}.$$

5. Relation to Schur functions:

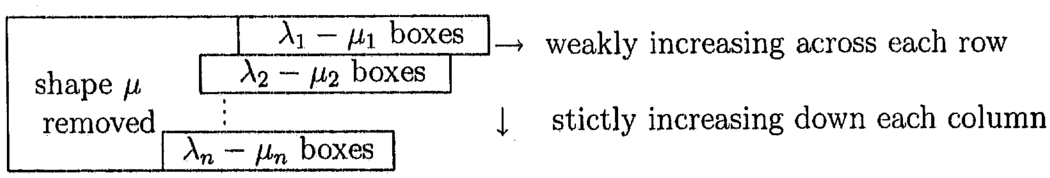
$$s_{\mu}(x) s_{\nu}(x) = \sum_{\lambda} c^{\lambda}_{\mu,\nu} s_{\lambda}(x),$$

$$Trace D^{\lambda}(diag(x_1, x_2, \ldots, x_n)) = s_{\lambda}(x)$$

6. Importance for physics: The matrices $D^{\lambda}(Z)$ give all inequivalent integer representations of $GL(n, C)$ for $Z \in GL(n, C)$. Thus, the matrices $D^{\lambda}(Z)$ also give all inequivalent unitary representations of $U(n)$ for $Z \in U(n)$. Moreover, these matrices and their elements occur as state vectors for composite physical systems and have a wealth of properties for various choices of the variables Z. We will see some of this in the following sections.

3 Skew Tableaus and Skew Gelfand-Tsetlin Patterns

Semistandard skew Young tableau: Let $\lambda, \mu \in \mathbb{P}ar_n$ with $\lambda_i \geq \mu_i$, $i = 1, 2, \ldots, n$. This condition is denoted $\lambda \supseteq \mu$, and means that the shape μ "fits inside" the shape λ. The *skew shape* $\lambda - \mu$ refers to the shape of the "staggered" rows of boxes that remain after deleting all the boxes of shape μ from the shape λ. As with a semistandard Young-Weyl tableau, the boxes in the shape $\lambda - \mu$ are filled in with the integers $1, 2, \ldots, n$ with one integer per box such that the sequence in each row is weakly increasing and the sequence in each column is strictly increasing:



Notations:

$$\mathbb{T}_{\lambda/\mu} = \text{set of all skew tableau of shape } \lambda - \mu,$$

$$\alpha = \text{weight of a skew tableau} = (\alpha_1, \ldots, \alpha_n), \alpha_i = \text{number of } i\text{'s },$$

$\mathbb{T}_{\lambda/\mu}(\alpha)$ = set of all skew tableau of shape $\lambda - \mu$ and weight α
$\mathbb{W}_{\lambda/\mu}(\alpha)$ = set of all weights in $\mathbb{T}_{\lambda/\mu}(\alpha)$,
$\mathbb{W}_{\lambda/\mu}$ = set of all weights in $\mathbb{T}_{\lambda/\mu}$,
$K_{\lambda/\mu}(\alpha)$ = multiplicity of weight $\alpha \in \mathbb{W}_{\lambda/\mu}$.

Skew Gelfand-Tsetlin patterns: The notation $\rho \sqsupseteq \sigma$ means that the pair of partitions $\rho,\ \sigma \in \mathbb{P}ar_n$ satisfy the betweenness conditions:

$$\rho_1 \geq \sigma_1 \geq \rho_2 \geq \sigma_2 \geq \cdots \geq \rho_n \geq \sigma_n \geq 0.$$

Two-rowed presentation:

$$\begin{pmatrix} \rho_1 & & \rho_2 & & \rho_3 & \cdots & \rho_{n-1} & & \rho_n & \\ & \sigma_1 & & \sigma_2 & & \cdots & & \sigma_{n-1} & & \sigma_n \end{pmatrix}.$$

A skew GT pattern of shape $\lambda - \mu$, with $\mu \subseteq \lambda$, is a sequence of $n+1$ partitions $\lambda, \tau, \ldots, \nu, \mu$ beginning with λ and ending with μ, all of which belong to $\mathbb{P}ar_n$, and which satisfy the conditions $\lambda \sqsupseteq \tau \sqsupseteq \cdots \sqsupseteq \nu \sqsupseteq \mu$. A skew GT pattern of shape $\lambda - \mu$ may be presented as a parallelogram with $n+1$ rows:

$$\begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_n & & & \\ & \tau_1 & \tau_2 & \cdots & \tau_n & & \\ & & & \vdots & & & \\ & & & \nu_1 & \nu_2 & \cdots & \nu_n \\ & & & & \mu_1 & \mu_2 & \cdots \ \mu_n \end{pmatrix},\ \lambda \sqsupseteq \tau \sqsupseteq \cdots \sqsupseteq \nu \sqsupseteq \mu.$$

The placement of the entries in this configuration is intended to be suggestive of the betweenness relations.

Notations:

$\mathbb{G}_{\lambda/\mu}$ = set of all skew GT patterns of shape $\lambda - \mu$,
α = weight of a skew GT pattern $= (\alpha_1, \ldots, \alpha_n)$
$= (|\nu| - |\mu|, \ldots, |\tau| - |\sigma|, |\lambda| - |\tau|)$,
$\mathbb{G}_{\lambda/\mu}(\alpha)$ = set of all skew GT patterns of shape $\lambda - \mu$ and weight α ,
$\mathbb{W}_{\lambda/\mu}(\alpha)$ = set of all weights in $\mathbb{G}_{\lambda/\mu}(\alpha)$,
$\mathbb{W}_{\lambda/\mu}$ = set of all weights in $\mathbb{G}_{\lambda/\mu}$,
$K_{\lambda/\mu}(\alpha)$ = multiplicity of weight $\alpha \in \mathbb{W}_{\lambda/\mu}$.

These notations for weights anticipate that these sets are identical to those defined for semistandard skew tableau.

Again, we have the bijection between the sets of semistandard skew tableaux and skew GT patterns of the same weight,

$$\mathbb{T}_{\lambda/\mu}(\alpha) \overset{BIJECTION}{\longleftrightarrow} \mathbb{G}_{\lambda/\mu}(\alpha),$$

as given by the following rules:

$$\begin{array}{l} \qquad\qquad \boxed{\lambda_1 - \mu_1 \text{ integers}} \\ \qquad \boxed{\lambda_2 - \mu_2 \text{ integers}} \\ \qquad \vdots \\ \quad \boxed{\lambda_n - \mu_n \text{ integers}} \end{array} \longleftrightarrow \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_n & & & \\ & \tau_1 & \tau_2 & \cdots & \tau_n & & \\ & & & \vdots & & & \\ & & & \kappa_1 & \kappa_2 & \cdots & \kappa_n \\ & & & & \nu_1 & \nu_2 & \cdots \ \nu_n \\ & & & & & \mu_1 & \mu_2 \ \cdots \ \mu_n \end{pmatrix}$$

mapping rule: The shapes in the skew GT pattern are obtained by sequential removal of integers from the skew tableau:

λ is the shape of the tableau;
τ is the shape of the tableau obtained by removing all n's;
$\vdots$
ν is the shape of the tableau obtained by removing all 2's;
μ is the shape of the tableau obtained by removing all 1's;

inverse rule: The skew tableau is obtained from the skew GT pattern by the rule

insert 1's in the shape $\nu - \mu$; followed by 2's in the shape $\kappa - \nu$; $\cdots$; followed by n's in the shape $\lambda - \tau$.

Example: $\lambda = (6530), \mu = (3210)$:

$$\begin{array}{|c|c|c|c|c|c|} \hline & & & 1 & 1 & 3 \\ \hline & & 1 & 3 & 4 & \\ \hline & 2 & 4 & & & \\ \hline \end{array} \quad \longleftrightarrow \quad \begin{array}{ccccccccc} 6 & & 5 & & 3 & & 0 & & \\ & 6 & & 4 & & 2 & & 0 & \\ & & 5 & & 3 & & 2 & & 0 \\ & & & 5 & & 3 & & 1 & & 0 \\ & & & & 3 & & 2 & & 1 & & 0 \end{array}$$

The bijection described above extends, of course, to the full sets of semi-standard skew Young tableaux and skew Gelfand-Tsetlin patterns of shape $\lambda - \mu$:

$$\mathbb{T}_{\lambda/\mu} \overset{BIJECTION}{\longleftrightarrow} \mathbb{G}_{\lambda/\mu},$$

4 The $D^{\lambda/\mu}$–Polynomials

It is somewhat surprising that the set of skew GT patterns $\mathbb{G}_{\lambda/\mu}$ can all be realized as ordinary triangular GT patterns[9] in $2n$ rows corresponding to partitions $(\lambda\, 0^n)$, $\lambda \in \mathbb{P}_n$, with further specialization of the patterns, as we describe in this section.

4.1 Relation between Skew GT Patterns and GT Patterns

We introduce the more detailed $m_{i,j}$ notation as follows for a skew GT pattern:

$$\begin{bmatrix} \lambda/\mu \\ m \end{bmatrix} = \begin{pmatrix} \lambda_1 \quad \lambda_2 \quad \cdots \quad \lambda_n \\ m_{1,n-1} \; m_{2,n-1} \; \cdots \; m_{n,n-1} \\ \vdots \\ m_{1,2} \quad m_{2,2} \quad \cdots \quad m_{n,2} \\ m_{1,1} \quad m_{2,1} \quad \cdots \quad m_{n,1} \\ \mu_1 \quad \mu_2 \quad \cdots \quad \mu_n \end{pmatrix},$$

in which

$$\lambda \sqsupseteq m_{n-1} \sqsupseteq \cdots \sqsupseteq m_1 \sqsupseteq \mu,$$

$$
\begin{aligned}
m_j &= (m_{1,j}, m_{2,j}, \dots, m_{n,j}) \in \mathbb{P}ar_n, j = 0, 1, \dots, n,\\
\lambda &= (m_{1,n}, m_{2,n}, \dots, m_{n,n}),\\
\mu &= (m_{1,0}, m_{2,0}, \dots, m_{n,0}).
\end{aligned}
$$

Consider next the triangular GT pattern with $2n$ rows and partition $(\lambda, 0^n), \lambda \in \mathbb{P}ar_n$ as follows:

$$
\left(\begin{array}{ccccccccccc}
\lambda_1 & \lambda_2 & \cdots & \lambda_n & 0 & \cdots & & & 0 \\
m_{1,n-1} & m_{2,n-1} & \cdots & m_{n,n-1} & 0 & \cdots & & 0 & \\
& \vdots & & & \vdots & & & & \\
& m_{1,2} & m_{2,2} & \cdots & m_{n,2} & 0 & 0 & & \\
& m_{1,1} & m_{2,1} & \cdots & m_{n,1} & 0 & & & \\
& \mu_1 & \mu_2 & \cdots & \mu_n & & & & \\
& & & & & & & & \\
& & & l & & & & &
\end{array}\right) = \left(\begin{array}{cc} \left[\begin{smallmatrix} \lambda/\mu \\ m \end{smallmatrix} \right] & (0) \\ \begin{pmatrix} \mu \\ l \end{pmatrix} & \end{array} \right),
$$

where $\binom{\mu}{l}$ is an ordinary triangular GT pattern with n rows. Notice that in this notation μ is included both as the bottom row in the skew pattern $\left[\begin{smallmatrix} \lambda/\mu \\ m \end{smallmatrix} \right]$ and as the top row in $\binom{\mu}{l}$. We then have the one-to-one correspondence between patterns in which we choose the entries $l_{i,j}$ in $\binom{\mu}{l}$ to be maximal, that is, $l_{i,j} = \mu_i, j = i, 2, \dots, n-1; i = 1, 2, \dots, n-1$.

$$
\left[\begin{smallmatrix} \lambda/\mu \\ m \end{smallmatrix} \right] \longleftrightarrow \left(\begin{array}{cc} \left[\begin{smallmatrix} \lambda/\mu \\ m \end{smallmatrix} \right] & (0) \\ \begin{pmatrix} \mu \\ max \end{pmatrix} & \end{array} \right).
$$

The following $D^{(\lambda\ 0^n)}$–polynomials, which are labeled by a pair of Gelfand-Tsetlin pattern having partition $(\lambda, 0^n), \lambda \in \mathbb{P}ar_n$, are fully defined, as discussed in Section 2.2:

$$
D\left(\begin{array}{cc} \begin{pmatrix} l' \\ \mu \end{pmatrix} & \\ \left[\begin{matrix} m' \\ \lambda/\mu \\ m \end{matrix} \right] & \langle 0 \rangle \\ \begin{pmatrix} \mu \\ l \end{pmatrix} & \end{array} \right)(Z_{2n}).
$$

We now define the following polynomials, which are labeled by double skew GT patterns, as the special case of these polynomials corresponding to choosing the patterns $\binom{\mu}{l}$ and $\binom{l'}{\mu}$ to be maximal:

$$D\begin{bmatrix} m' \\ \lambda/\mu \\ m \end{bmatrix}(Z) = D\left(\begin{array}{c} \begin{pmatrix} max \\ \mu \end{pmatrix} \\ \\ \begin{bmatrix} m' \\ \lambda \\ m \end{bmatrix} \langle 0 \rangle \\ \\ \begin{pmatrix} \mu \\ max \end{pmatrix} \end{array}\right) \begin{pmatrix} I_n & 0 \\ 0 & Z \end{pmatrix},$$

with weights of lower and upper patterns given by

$$\gamma = (\mu, \alpha),\ \ \alpha = \text{weight of } \begin{bmatrix} \lambda/\mu \\ m \end{bmatrix},$$

$$\gamma' = (\mu, \alpha'),\ \ \alpha' = \text{weight of } \begin{bmatrix} \lambda/\mu \\ m' \end{bmatrix}.$$

These polynomials then have the following properties, as expressed in terms of the matrices of dimension

$$Dim D^{\lambda/\mu}(Z) = \sum_{\alpha \in \mathbf{W}_{\lambda/\mu}} K(\lambda/\mu, \alpha),$$

which are obtained by using the lower patterens $\begin{bmatrix} \lambda/\mu \\ m \end{bmatrix}$ to label rows and the upper patterns $\begin{bmatrix} m' \\ \lambda/\mu \end{bmatrix}$ to label columns:

1. Multiplication of $D^{\lambda/\mu}(Z)$ matrices:

$$D^{\lambda/\mu}(X) D^{\lambda/\mu}(Y) = D^{\lambda/\mu}(XY),$$

   for arbitrary matrices X and Y. This multiplication property is a direct consequence of the definition of these matrices, the multiplication property for ordinary $D^{\lambda}(Z)$ matrices, and the fact that the restriction to maximal labels propagates through products.

2. Matrix skew Schur function:

$$D^{\lambda/\mu}(Z) \sim \sum_{\nu} c^{\lambda}_{\mu\nu} D^{\nu}(Z).$$

3. Diagonal properties:

$$D^{\lambda/\mu}(diag(x_1, x_2, \ldots, x_n)) = \sum_{\alpha \in \mathbf{W}_{\lambda/\mu}} \oplus x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} I_{K(\lambda/\mu,\alpha)},$$

$$I_{K(\lambda/\mu,\alpha)} = \text{ identity matrix of dimension } K(\lambda/\mu, \alpha),$$

$$D^{\lambda/\mu}(I_n) = I_{Dim\lambda/\mu}.$$

4. Relation to skew Schur functions:

$$s_{\lambda/\mu}(x) = \sum_{\nu} c^{\lambda}_{\mu,\nu} s_{\nu}(x),$$

$$Trace D^{\lambda/\mu}(diag(x_1, x_2, \ldots, x_n)) = s_{\lambda/\mu}(x)$$

5. Importance for physics: The matrices $D^{\lambda/\mu}(Z)$ give a new class of reducible integer representations of $GL(n, C)$ for $Z \in GL(n, C)$, and, similarly, for $U(n)$ by choosing $Z \in U(n)$. We expect to uncover a wealth of other properties for various choices of the variables Z.

5 Words of Semistandard Skew Tableau and Skew GT Patterns

The notion of a word of a words of semistandard skew tableau and skew GT patterns is equivalent to that of weight, but the former is more closely related to the notion of a lattice permutation. The enumeration of lattice permutations of a certain type gives the Littlewood-Richardson numbers $c^{\lambda}_{\mu,\nu}$, which, in turn, among other occurrences, are important quantities for reducing Kronecker products of irreducible representations of $U(n)$. In this section, we give some basic definitions and properties of words, as follows:

The standard form of a sequence of repeated $1's, \ldots, n's$: Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ be a sequence of non-negative integers with $|\alpha| = k$. The weakly increasing sequence defined by

$$(1, 2, \cdots, n)^{\alpha} = 1^{\alpha_1} 2^{\alpha_2} \cdots n^{\alpha_n}$$

is called the *standard form* or *type* of any sequence

$$A_k = a_1 a_2 \ldots a_k, \;\; \alpha_1 + 2\alpha_2 + \cdots + n\alpha_n = k$$

containing α_1 $1's, \alpha_2$ $2's, \ldots, \alpha_n$ $n's$. We denote the set of all such sequences by $\mathbb{A}_k(\alpha)$, which then has cardinality given by the multinomial coefficient:

$$|\mathbb{A}_k(\alpha)| = \binom{k}{\alpha_1, \alpha_2, \ldots, \alpha_n}.$$

5.1 Word of a Semistandard Skew Tableau of Shape $\lambda - \mu$

Row i of the semistandard skew tableau

$\boxed{\lambda_1 - \mu_1 \text{ integers}}$ $\longrightarrow$ weakly increasing across each row

shape μ $\boxed{\lambda_2 - \mu_2 \text{ integers}}$

removed $\vdots$ $\downarrow$ stictly increasing down each column

$\boxed{\lambda_n - \mu_n \text{ integers}}$

is filled out as follows:

$$\begin{array}{cccccccccccc} \multicolumn{3}{c}{l_{i,1}\ 1's} & \multicolumn{3}{c}{l_{i,2}\ 2's} & & \multicolumn{3}{c}{l_{i,n}\ n's} & \\ \boxed{1} & \cdots & \boxed{1} & \boxed{2} & \cdots & \boxed{2} & \cdots & \boxed{n} & \cdots & \boxed{n} & \lambda_i - \mu_i \end{array}$$

Then, by reverse (right-to-left) reading of row i we obtain the sequence

$$L^{(i)}_{\lambda/\mu} = n^{l_{in}} \cdots 2^{l_{i2}} 1^{l_{i1}}.$$

The *word* of this semistandard skew tableau of weight α is the sequence defined by

$$L_{\lambda/\mu}(\ell) = L^{(1)}_{\lambda/\mu} L^{(2)}_{\lambda/\mu} \cdots L^{(n)}_{\lambda/\mu},$$

where the weight α is given by

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n), \ \alpha_j = l_{1,j} + l_{2,j} + \ldots + l_{n,j}, j = 1, 2, \ldots, n.$$

This word sequence is of type $(1, 2, \ldots, n)^\alpha$. The n^2 non-negative integers

$$\ell = (l_{ij})_{1 \le i,j \le n}$$

appearing in this word must, of course, fulfill the rules for a semistandard skew tableau.

5.2 Word of a Skew Gelfand-Tsetlin Pattern of Shape $\lambda - \mu$

Consider again the skew GT pattern

$$\begin{bmatrix} \lambda/\mu \\ m \end{bmatrix} = \begin{array}{cccc} \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ & & \vdots & \\ m_{1,j-1} & m_{2,j-1} & \cdots & m_{n,j-1} \\ & & \vdots & \\ m_{1,1} & m_{2,1} & \cdots & m_{n,1} \\ \mu_1 & \mu_2 & \cdots & \mu_n \end{array},$$

which has weight

$$\alpha = (|m_1| - |m_0|, |m_2| - |m_1|, \ldots, |m_n| - |m_{n-1}|).$$

The $i-$th left diagonal ($i = 1, 2, \ldots, n$) of this pattern is mapped to the sequence $L^{(i)}_{\lambda/\mu}$, that is,

$$\begin{array}{ccc} m_{i,n} & & \\ & \ddots & \\ & m_{i,1} & \\ & & m_{i,0} \end{array} \longrightarrow L^{(i)}_{\lambda/\mu} = n^{l_{in}} \cdots 2^{l_{i2}} 1^{l_{i1}},$$

where the non-negative integers $l_{i,j} = m_{i,j} - m_{i,j-1}, j = 1, 2, \ldots, n$, are read as successive differences along this diagonal. This sequence of type $(1, 2, \ldots, n)^\alpha$ defines the *word* of this skew pattern:

$$L \begin{bmatrix} \lambda/\mu \\ m \end{bmatrix} = L^{(1)}_{\lambda/\mu} L^{(2)}_{\lambda/\mu} \cdots L^{(n)}_{\lambda/\mu}.$$

Since the bijection

$$\mathbb{T}_{\lambda/\mu} \overset{BIJECTION}{\longleftrightarrow} \mathbb{G}_{\lambda/\mu}$$

is exactly the one-to-one correspondence given by setting $l_{i,j} = m_{i,j} - m_{i,j-1}$, $j = 1, 2, \ldots, n$ in row i of the semistandard skew tableau, each $i = 1, 2, \ldots, n$, the words of $\mathbb{T}_{\lambda/\mu}$ and $\mathbb{G}_{\lambda/\mu}$ are all identical:

$$L_{\lambda/\mu}(\ell) = L \left[\begin{array}{c} \lambda/\mu \\ m \end{array} \right], \text{for all patterns}$$

Notations:

$\mathbb{L}_{\lambda/\mu}(\alpha)$ = set of all words of type $(1, 2, \ldots, n)^\alpha$ of semistandard skew tableaux or skew GT patterns of shape $\lambda - \mu$,

$\mathbb{L}_{\lambda/\mu}$ = set of all words of semistandard skew tableaux or skew GT patterns of shape $\lambda - \mu$.

6 Lattice Permutations and Littlewood-Richardson Numbers

The properties of the Littlewood-Richardson numbers $c^{\lambda}_{\mu\nu}$ are basic to the reduction of Kronecker products of D^{λ}−polynomials and to the reduction of skew $D^{\lambda/\mu}$−polynomials, which is one of the reasons we study them. Their relation to lattice permutations is one such fundamental property. First, we define a lattice permutation.

6.1 Lattice Permutations A word

$$A_k = a_1 a_2 \ldots a_j \ldots a_k \in \mathbb{A}_k(\alpha)$$

of type $(1, 2, \ldots, n)^\alpha$ is a lattice permutation if and only if Rule L as follows is true:

> Rule L : In each left factor $A_j = a_1 a_2 \ldots a_j, 1 \leq j \leq k$, of A_k the number of $i's$ is greater than or equal to the number of $i + 1's$

It follows from this rule that the sequence α is a **partition,** which we henceforth denote by ν.

From now on, we develop all further results in the language of skew Gelfand-Tsetlin patterns , although all results could be rephrased in the language of semistandard skew tableau.

*Notations:*Let $\lambda, \mu, \nu \in \mathbb{P}ar_n$ with $\mu, \nu \subseteq \lambda, |\lambda| = |\mu| + |\nu|$. Define the following sets:

$$\mathbb{G}_{\lambda/\mu}(\nu) = \left\{ \text{subset of } \mathbb{G}_{\lambda/\mu} \text{ of weight } \nu \right\},$$

$\mathbb{G}^{\lambda}_{\mu,\nu} = \{\text{subset of } \mathbb{G}_{\lambda/\mu}(\nu) \text{ having words that are lattice permutations }\}$,
$\mathbb{L}_{\lambda/\mu}(\nu) = \{\text{set of words corresponding to all patterns in } \mathbb{G}_{\lambda/\mu}(\nu)\ \}$
$\mathbb{L}^{\lambda}_{\mu,\nu} = \{\text{set of lattice permutations corresponding to all patterns in } \mathbb{G}^{\lambda}_{\mu,\nu}\}$

5.2 Littlewood-Richardson Numbers

Let $\lambda, \mu, \nu \in \mathbb{P}ar_n$. The Littlewood-Richardson number $c^{\lambda}_{\mu\nu}$ are given by

$$c^{\lambda}_{\mu\nu} = \begin{cases} |\mathbb{L}^{\lambda}_{\mu,\nu}|, & \text{for } \mu, \nu \subseteq \lambda,\ |\lambda| = |\mu| + |\nu| \\ 0, \text{otherwise} \end{cases}$$

(See Littlewood and Richardson,[10] Macdonald,[11] Stanley.[7]) Thus, a basic problem is:

Find the set of words $\mathbb{L}^{\lambda}_{\mu,\nu}$.

We shall not solve this problem here, but we can find a subset of $\mathbb{G}_{\lambda/\mu}(\nu)$ that contains $\mathbb{L}^{\lambda}_{\mu,\nu}$, and this narrows considerably the problem. For this, we make the following two observations:

Observation 1: A skew GT pattern can be split by the minor diagonal of the parallelogram into two triangular patterns as follows:

$$\begin{bmatrix} \lambda/\mu \\ m \end{bmatrix} = \left[\begin{pmatrix} \lambda \\ m' \end{pmatrix} \Big/ \begin{pmatrix} m'' \\ \mu \end{pmatrix}\right]$$

Examples for $n = 2, 3$:

$$\begin{array}{cccc} \lambda_1 & & \lambda_2 \,/ & \\ & m_{1,1} \,/ & & m_{2,1} \\ & / \mu_1 & & & \mu_2 \end{array} \qquad \begin{array}{ccccccc} \lambda_1 & & \lambda_2 & & \lambda_3 \,/ & & \\ & m_{1,2} & & m_{2,2} \,/ & & m_{3,2} & \\ & & m_{1,1} \,/ & & m_{2,1} & & m_{3,1} \\ & & / \mu_1 & & \mu_2 & & \mu_3 \end{array}$$

The pattern $\binom{\lambda}{m'}$ is a normal triangular GT pattern with n rows in which $m'_{i,j} = m_{i,j}, i = 1, 2, \ldots, n; j = 1, 2, \ldots, i$. The pattern $\binom{m''}{\lambda}$ is a normal inverted triangular GT pattern with n rows in which

$$m''_{i,j} = m_{n-j+i,n-j}, j = 1, 2, \ldots, n; i = 1, 2, \ldots, j.$$
$$\mu = (m'_{1,n}, m'_{2,n}, \ldots, m'_{n,n}).$$

Of course, the entries in these two triangular patterns are constrained by the betweenness relations for the full skew pattern.

Observation 2: For word of a skew GT pattern to be a lattice permutation, it is necessary that the pattern $\binom{\lambda}{m'}$ be maximal, that is,

$$\begin{array}{ccccccc} \lambda_1 & & \lambda_2 & \cdots & & & \lambda_n \\ & \lambda_1 & & \lambda_2 & \cdots & \lambda_{n-1} & \\ & & & \vdots & & & \\ & & \lambda_1 & & \lambda_2 & & \\ & & & \lambda_1 & & & \end{array} \qquad \text{maximal pattern}$$

*Proof.*The word of a skew GT pattern is

$$L\begin{bmatrix} \lambda/\mu \\ m \end{bmatrix} = L_{\lambda/\mu} = L^{(1)}_{\lambda/\mu} L^{(2)}_{\lambda/\mu} \cdots L^{(n)}_{\lambda/\mu}.$$

In order that this word be a lattice permutation, the following conditions are necessary:

$L^{(1)}_{\lambda/\mu} = n^{l_{1,n}} \cdots 2^{l_{1,2}} 1^{l_{1,1}}$ can have no entries to the left of $1^{l_{1,1}}$; hence,

$$l_{1,n} = l_{1,n-1} = l_{1,2} = 0$$

$L^{(2)}_{\lambda/\mu} = n^{l_{2,n}} \cdots 2^{l_{2,2}} 1^{l_{2,1}}$ can have no entries to the left of $2^{l_{2,2}}$; hence,

$$l_{2,n} = l_{2,n-1} = l_{2,3} = 0$$

$\vdots$

$L^{(i)}_{\lambda/\mu} = n^{l_{i,n}} \cdots 2^{l_{i,2}} 1^{l_{i,1}}$ can have no entries to the left of $i^{l_{i,i}}$; hence,

$$l_{i,n} = l_{i,n-1} = l_{i,i+1} = 0$$

$\vdots$

$L^{(n)}_{\lambda/\mu} = n^{l_{n,n}} \cdots 2^{l_{n,2}} 1^{l_{n,1}}$ can have no entries to the left of $n^{l_{n,n}}$; hence,

no condition

This collection of conditions

$$l_{ij} = 0, \quad \text{equivalently, } m'_{i,j-1} = m'_{i,j} j = i+1, i+2, \cdots, n; i = 1, 2, \cdots, n-1$$

are just the conditions that the pattern $\binom{\lambda}{m'}$ be maximal, so that

$$\begin{bmatrix} \lambda/\mu \\ m \end{bmatrix} = \left[\begin{pmatrix} \lambda \\ max \end{pmatrix} \Big/ \begin{pmatrix} m'' \\ \mu \end{pmatrix}\right], \square$$

The word of this special skew GT pattern is

$$L\left[\begin{pmatrix} \lambda \\ max \end{pmatrix} \Big/ \begin{pmatrix} m'' \\ \mu \end{pmatrix}\right]$$

$$= 1^{l_{1,1}} \qquad 2^{l_{2,2}}\, 1^{l_{2,1}} \qquad 3^{l_{3,1}}\, 2^{l_{3,2}}\, 1^{l_{3,3}} \qquad \cdots \qquad n^{l_{n,n}} \cdots 2^{l_{n,2}}\, 1^{l_{n,1}}.$$

This result can be put in better form: The information is fully encoded in the inverted pattern $\binom{m''}{\mu}$ and the boundary right diagonal $(\lambda_1, \lambda_2, \cdots, \lambda_n)$

of the pattern $\binom{\lambda}{max}$. We rearrange as follows: The inverted pattern is put in normal un inverted form, keeping the $\lambda-$diagonal, and removing the double prime from $m''_{i,j}$ to arrive at the *modified Gelfand-Tsetlin pattern* given by

$$\begin{pmatrix} \lambda/\mu \\ m \end{pmatrix} = \begin{array}{cccccc} \emptyset & \mu_1 & \mu_2 & \mu_3 & \cdots & \mu_n \\ \lambda_1 & m_{1,n-1} & m_{2,n-1} & \cdots & m_{n-1,n-1} & \\ \lambda_2 & m_{1,n-2} & m_{2,n-2} \cdots m_{n-2,n-2} & & & \\ \vdots & & & & & \\ \lambda_{n-1} & m_{1,1} & & & & \\ \lambda_n. & & & & & \end{array},$$

in which $\emptyset$ designates that no entry is placed in that position. The word above is now given by

$$L\begin{pmatrix} \lambda/\mu \\ m \end{pmatrix} = \underbrace{1^{r_{1,1}}} \; \underbrace{2^{r_{2,2}}\,1^{r_{2,1}}} \cdots \underbrace{i^{r_{i,i}} \ldots 2^{r_{i,2}}\,1^{r_{i,1}}} \cdots \underbrace{n^{r_{n,n}} \ldots 2^{r_{n,2}}\,n^{r_{n,1}}},$$

where the exponents of the $i-$th partial word are read off the $i-$th right diagonal of the pattern:

$$\begin{array}{l} \mu_i = m_{i,n} \\ \quad m_{i-1,n-1} \\ m_{i-2,n-2} \\ \vdots \\ m_{1,n-i+1} \\ \lambda_i = m_{0,n-i} \end{array} \longrightarrow i^{r_{i,i}} \ldots 2^{r_{i,2}}\,1^{r_{i,1}},$$

where the

$$r_{i,j} = m_{i-j+1,n-j+1} - m_{i-j,n-j},\; j = i, i-1, \ldots, 1,$$

are the successive difference between the entries along the diagonal.

The weight of a modified Gelfand-Tselin pattern is defined by

$$\alpha = W\begin{pmatrix} \lambda/\mu \\ m \end{pmatrix} = (\alpha_1, \alpha_2, \ldots, \alpha_n), \alpha_i = r_{i,i} + r_{i+1,i} + \ldots + r_{n,i},$$

in which α_i denotes the number of $i's$ that appear in the word $L\binom{\lambda/\mu}{m}$. Thus, the word $L\binom{\lambda/\mu}{m}$ is of type $(1, 2, \ldots, n)^{\alpha}$.

Given $\mu, \lambda \in \mathbb{P}ar_n$ with $\mu \subseteq \lambda$, the entries in a modified Gelfand-Tsetlin patterns satisfy all the standard betweenness relations, and the presence of $\emptyset$ serves only to fill out the triangular array, and be a reminder that λ_1 may be as large as one pleases.

Notations:

$$M\mathbb{L}_{\lambda/\mu} = \Big\{ \text{set of words } L\tbinom{\lambda/\mu}{m} \;\Big|\; m \text{ runs over all values for which } \tbinom{\lambda/\mu}{m} \text{ is a modified } GT \text{ pattern} \Big\}$$

$$M\mathbb{L}_{\lambda/\mu}(\nu) = \left\{\text{subset of } M\mathbb{L}_{\lambda/\mu} \,\middle|\, W\begin{pmatrix}\lambda/\mu \\ m\end{pmatrix} = \nu \supseteq \lambda\right\}$$
$$M\mathbb{L}^{\lambda}_{\mu,\nu} = \left\{\text{subset of words in } M\mathbb{L}_{\lambda/\mu}(\nu) \text{ that are lattice permutations}\right\}.$$

By design, a modified GT pattern captures all those words in $\mathbb{L}_{\lambda/\mu}(\nu)$ that can possibly be lattice permutations, that is, $M\mathbb{L}^{\lambda}_{\mu,\nu} = \mathbb{L}^{\lambda}_{\mu,\nu}$. Accordingly, we have that, for $\mu, \nu, \lambda \in \mathbb{P}ar_n$ with $\mu, \nu \subseteq \lambda$ and $|\lambda| = |\mu| + |\nu|$, then

$$c^{\lambda}_{\mu\,\nu} = |M\mathbb{L}^{\lambda}_{\mu,\nu}| = |\mathbb{L}^{\lambda}_{\mu,\nu}|.$$

Examples:

n=2:

$$\begin{array}{ccccc} \emptyset & & \mu_1 & & \mu_2 \\ & \lambda_1 & & m_{1,1} & \\ & & \lambda_2 & & \end{array}$$

$$\text{word} = 1^{\lambda_1-\mu_1}\ 2^{\lambda_2-m_{1,1}}\ 1^{m_{1,1}-\mu_2}$$

word = lattice permutation if and only if $\lambda_1 - \mu_1 \geq \lambda_2 - m_{1,1}$.

Recall the familiar angular momentum addition rule as given by the Clebsch-Gordan series

$$(2j_1\,0)\otimes(2j_2\,0) = \sum_{j=j_1-j_2}^{j_1+j_2} \oplus(j_1+j_2+j, j_1+j_2-j), \text{ for } j_1 \geq j_2$$

for which $\mu = (2j_1\,0), \nu = (2j_2\,0), \lambda = (j_1+j_2+j, j_1+j_2-j)$. The unique modified GT pattern of weight $\nu = (2j_2\,0)$ is

$$\begin{array}{ccccc} \emptyset & & 2j_1 & & 0 \\ & j_1+j_2+j & & j_1+j_2-j & \\ & & j_1+j_2-j & & \end{array} \quad \mapsto \quad \text{word} = 1^{j_2-j_1+j}\ 2^0\ 1^{j_1+j_2-j} = 1^{2j_2}$$

Thus,the values of the total angular momentum j are *exactly* those for which the GT pattern satisfies betweenness, each such j yielding the same lattice permutation of weight $(2j_2\,0)$. We conclude:

$$c^{(j_1+j_2+j, j_1+j_2-j)}_{(2j_1\,0),(2j_2\,0)} = \begin{cases} 1, \text{each } j = j_1-j_2, j_1-j_2+1, \ldots, j_1+j_2; \\ 0, \text{otherwise} \end{cases}$$

n=3

$$\begin{array}{ccccccc} \emptyset & & \mu_1 & & \mu_2 & & \mu_3 \\ & \lambda_1 & & m_{1,2} & & m_{2,2} & \\ & & \lambda_2 & & m_{1,1} & & \\ & & & \lambda_3 & & & \end{array}$$

$$\text{word} = 1^{\lambda_1-\mu_1} 2^{\lambda_2-m_{1,2}} 1^{m_{1,2}-\mu_2} 3^{\lambda_3-m_{1,1}} 2^{m_{1,1}-m_{2,2}} 1^{m_{2,2}-\mu_3}.$$

$$\text{weight} = \nu = (\lambda_1 - \mu_1 + m_{1,2} - \mu_2 + m_{2,2} - \mu_3, \quad \lambda_2 - m_{1,2} + m_{1,1} - m_{2,2}, \lambda_3 - m_{1,1}).$$

Numerical example 1: $\mu = (4,2,0), \nu = (5,4,1), \lambda = (7,6,3)$. There are three modified GT patterns of type $(1,2,3)^{(5,4,1)} = 1^5\,2^4\,3^1$:

$$\begin{array}{ccccccc} \emptyset & & 4 & & 2 & & 0 \\ & 7 & & 2 & & 2 & \\ & & 6 & & 2 & & \\ & & & 3 & & & \end{array} \quad \mapsto \quad 1^3\,2^4\,3^1\,1^2 \quad \text{(non lattice)}$$

$$\begin{array}{ccccccc} \emptyset & & 4 & & 2 & & 0 \\ & 7 & & 3 & & 1 & \\ & & 6 & & 2 & & \\ & & & 3 & & & \end{array} \quad \mapsto \quad 1^3\,2^3\,1^1\,3^1\,2^1\,1^1 \quad \text{(lattice)}$$

$$\begin{array}{ccccccc} \emptyset & & 4 & & 2 & & 0 \\ & 7 & & 4 & & 0 & \\ & & 6 & & 2 & & \\ & & & 3 & & & \end{array} \quad \mapsto \quad 1^3\,2^2\,1^2\,3^1\,2^2 \quad \text{(lattice)}$$

Therefore: $c^{(7\,6\,3)}_{(4\,2\,0),(5\,4\,1)} = 2$.

Numerical example 2: $\mu = (2,2,2), \nu = (5,4,1), \lambda = (7,6,3)$. There is one modified GT patterns of type $(1,2,3)^{(5,4,1)} = 1^5\,2^4\,3^1$:

$$\begin{array}{ccccccc} \emptyset & & 2 & & 2 & & 2 \\ & 7 & & 2 & & 2 & \\ & & 6 & & 2 & & \\ & & & 3 & & & \end{array} \quad \mapsto \quad 1^5\,2^4\,3^1 \quad \text{(lattice)}$$

Therefore: $c^{(7\,6\,3)}_{(2\,2\,2),(5\,4\,1)} = 1$.

It is tractable to carry the above forward and derive necessary and sufficient conditions that a pattern in the set of modified GT patterns corresponds to a lattice permutation of given weight. This is important for the theory of tensor operators in $U(n)$.

The $D-$polynomials corresponding to modified GT patterns can also be given, since they occur as specializations of the general polynomials discussed in Section :

$$D\begin{pmatrix} l' \\ \lambda/\mu \\ l \end{pmatrix}(Z_{2n}) = D\begin{pmatrix} & \binom{l'}{\mu} & \\ \begin{bmatrix} max \\ \lambda/\mu \\ max \end{bmatrix} & & \langle 0 \rangle) \\ & \binom{\mu}{l} & \end{pmatrix}(Z_{2n}) = D\begin{pmatrix} & l' & \\ & \mu & \\ \lambda & & 0 \\ & \mu & \\ & l & \end{pmatrix}(Z_{n+1}).$$

The dependence of these polynomials on only those variables occurring in the $(n+1)\times(n+1)$ submatrix Z_{n+1} of the $2n\times 2n$ matrix Z_{2n} is a consequence of the choice of maximal labels im the skew pattern $\begin{bmatrix} max \\ \lambda/\mu \\ max \end{bmatrix}$, which gives the weights of the lower and upper patterns as

$$\gamma = (\alpha, 0^n), \quad \gamma' = (\alpha', 0^n),$$
$$\alpha = W\begin{pmatrix} \mu \\ l \end{pmatrix}, \quad \alpha' = W\begin{pmatrix} \mu \\ l' \end{pmatrix}.$$

We have not yet worked out the properties of these polynomials labeled by pairs of modified Gelfand-Tsetlin patterns.

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